

n -QUIVERS AND A UNIVERSAL INVESTIGATION OF n -REPRESENTATIONS OF QUIVERS

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ABSTRACT. We have two parallel goals of this paper. First, we investigate and construct cofree coalgebras over n -representations of quivers, limits and colimits of n -representations of quivers, and limits and colimits of coalgebras in the monoidal categories of n -representations of quivers. Second, we introduce a generalization for quivers and show that this generalization can be seen as essentially the same as n -representations of quivers. This is significantly important because it allows us to build a new quiver $\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)}$, called n -quiver, from any given quivers Q_1, Q_2, \dots, Q_n , and enable us to identify each category $Rep_k(Q_j)$ of representations of a quiver Q_j as a full subcategory of the category $Rep_k(\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)})$ of representations of $\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)}$ for every $j \in \{1, 2, \dots, n\}$.

1. INTRODUCTION

The quiver representations theory has been developed exponentially since it was introduced in [16] due to Gabriel in 1972. Very recently, it has significantly invaded a large area of substantial studies, and it has been seen as a pivot source of inspiration for many theories. Due to its combinatorial flavor, it has become as a vital subject with germane connections to associative algebra, combinatorics, algebraic topology, algebraic geometry, quantum groups, Hopf algebras, tensor categories. As it is well-known, there is a forgetful functor, which has a left adjoint, from the category of small categories to the category of quivers. It turns out that the quiver representations theory supports a very positive connection between combinatorics and category theory, and this allows to substantially distinctive theories to collaboratively work together as a team to reshape many theories and build them in terms of modern techniques.

As an important strategy for developing quiver representations theory, we need to carefully and systematically generalize the notions of quivers and their representations. Our main aim is to introduce generalizations for quivers and their representations. Unlike one might expect, our generalization approach is to start with introducing a generalization for representations of quivers. Then we investigate universal properties and constructions for n -representations of quivers. Finally, we introduce a generalization for quivers and prove that this generalization and our previous generalization approach are essentially the same in the sense of a categorical point of view.

The first part of our aim has already been done in [3] by introducing the concept of n -representations of quivers, providing concrete examples of such notions, establishing categories of n -representations of quivers, and showing that these categories are abelian. For any $n \geq 2$, an n -representation of an n -tuple (Q_1, Q_2, \dots, Q_n) of quivers is a generalization of a representation of Q_j , for any $j \in \{1, 2, \dots, n\}$. Further, the definition of n -representations of quivers allows us to identify each category $Rep_k(Q_j)$ of representations of a quiver Q_j as a full subcategory of the category

Key words and phrases. Quiver, representation, birepresentation, n -representation, n -quiver, coalgebra, limit, colimit, complete, cocomplete, free, cofree, Monoidal category, generator, co-wellpowered, abelian category.

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$Rep_{(Q_1, Q_2, \dots, Q_n)}$ of n -representations of (Q_1, Q_2, \dots, Q_n) for every $j \in \{1, 2, \dots, n\}$.

This paper is mainly devoted to two parallel goals. The first one is to investigate cofree coalgebras over n -representations of quivers, limits and colimits of n -representations of quivers, and limits and colimits of coalgebras in the categories of n -representations of quivers. Moreover, we explicitly construct cofree coalgebras, limits, colimits in these (monoidal) categories.

The other goal of this paper is to introduce a generalization for quivers and prove that this generalization can be recast into n -representations of quivers. This is crucially important since for any given quivers Q_1, Q_2, \dots, Q_n , it allows us to build a new quiver $\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)}$, called n -quiver, by which we are able to view each category $Rep_k(Q_j)$ of representations of a quiver Q_j as a full subcategory of the category $Rep_k(\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)})$ of representations of $\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)}$ for every $j \in \{1, 2, \dots, n\}$.

To formulate our goal concerned with a universal investigation, we need to recall some categorical definitions.

Let \mathfrak{X} be a category. A *concrete category* over \mathfrak{X} is a pair $(\mathfrak{A}, \mathfrak{U})$, where \mathfrak{A} is a category and $\mathfrak{U} : \mathfrak{A} \rightarrow \mathfrak{X}$ is a faithful functor [4, p. 61]. Let $(\mathfrak{A}, \mathfrak{U})$ be a concrete category over \mathfrak{X} . Following [4, p. 140-143], a *free object* over \mathfrak{X} -object X is an \mathfrak{A} -object A such that there exists a *universal arrow* (A, u) over X ; that is, $u : X \rightarrow \mathfrak{U}A$ such that for every arrow $f : X \rightarrow \mathfrak{U}B$, there exists a unique morphism $f' : A \rightarrow B$ in \mathfrak{A} such that $\mathfrak{U}f'u = f$. We also say that (A, u) is the free object over X . A concrete category $(\mathfrak{A}, \mathfrak{U})$ over \mathfrak{X} is said to *have free objects* provided that for each \mathfrak{X} -object X , there exists a universal arrow over X . For example, the category $Vect_{\mathbb{K}}$ of vector spaces over a field \mathbb{K} has free objects. So do the category **Top** of topological spaces and the category of **Grp** of groups. However, some interesting categories do not have free objects [4, p. 142]).

Dually, *co-universal arrows*, *cofree objects* and categories that *have cofree objects* can be defined. For the basic concepts of concrete categories, free objects, and cofree objects, we refer the reader to [17, p. 138-155].

It turns out that a concrete $(\mathfrak{A}, \mathfrak{U})$ over \mathfrak{X} has (co)free objects if and only if the functor that builds up (co)free object is a (right) left adjoint to the faithful functor $\mathfrak{U} : \mathfrak{A} \rightarrow \mathfrak{X}$.

Therefore, our goal involved with a universal investigation can explicitly be formulated as follows. Let $\mathcal{U}_n : CoAlg(Rep_{(Q_1, Q_2, \dots, Q_n)}) \rightarrow Rep_{(Q_1, Q_2, \dots, Q_n)}$ be the forgetful functor from the category of coalgebras in the category $Rep_{(Q_1, Q_2, \dots, Q_n)}$ to the category $Rep_{(Q_1, Q_2, \dots, Q_n)}$. Does \mathcal{U}_n have a right adjoint?

An expected strategy for the answer of this question is to use the dual of Special Adjoint Functor Theorem (D-SAFT).

We start our inspection by showing that the limits and colimits in the categories of n -representations of quivers can be inherited from their respective limits and colimits of representations of quivers. Then we introduce the notion of n -quivers which suggests that there is an equivalence of categories between the categories of n -representations of quivers and the categories of representations of n -quivers. Eventually, we show that cofree coalgebras exist in the monoidal categories of n -representations of quivers and describe them in terms of cofree coalgebras in the monoidal categories of representations of quivers.

The sections of this paper can be summarized as follows.

In Section 2, we give some detailed background on quiver representations and few categorical notions that we need for the next sections. As a preliminary step, we briefly discuss the basics about n -representations of quivers. For more details and concrete examples of n -representations of quivers, we refer the reader to [3].

In Section 3, we briefly discuss the basics about n -representations of quivers.

In Section 4, we start with an investigation of limits of birepresentations (2-representations of quivers) and inductively extend our results to limits of n -representations of quivers. Similarly to the case of limits, we investigate and construct colimits of n -representations of quivers.

In Section 5, we introduce the notion of 2-quivers and inductively generalize the notion for of n -quivers. We explicitly give concrete examples of n -quivers and representations of n -quivers. Further, we show that the concept of n -quivers gives rise to identify the categories of n -representations of quivers and the categories of representations of n -quivers as essentially the same. This identification allows one to have an explicit description for the generators of the categories of n -representations of quivers and characterize many properties of n -representations of quivers.

Finally, we investigate cofree coalgebras in the monoidal categories of n -representations of quivers. We also construct them in terms of colimits and generators and show that cofree coalgebras in these monoidal categories can explicitly be obtained from cofree coalgebras in the monoidal categories of quiver representations.

2. Preliminaries

Throughout this paper k is an algebraically closed field, $n \geq 2$, and $Q, Q', Q_1, Q_2, \dots, Q_n$ are quivers. We also denote kQ the path algebra of Q . Unless otherwise specified, we will consider only finite, connected, and acyclic quivers.

Let \mathcal{A} be a (locally small) category and A, B objects in \mathcal{A} . We denote by $\mathcal{A}(A, B)$ the set of all morphisms from A to B .

Let \mathcal{A}, \mathcal{B} be categories. Following [21, p. 74], the **product category** $\mathcal{A} \times \mathcal{B}$ is the category whose objects are all pairs of the form (A, B) , where A is an object of \mathcal{A} and B an object of \mathcal{B} . An arrow is a pair $(f, g) : (A, B) \rightarrow (A', B')$, where $f : A \rightarrow A'$ is an arrow of \mathcal{A} and $g : B \rightarrow B'$ is an arrow of \mathcal{B} . The identity arrow for $\mathcal{A} \times \mathcal{B}$ is (id_A, id_B) and composition is defined component-wise, so $(f, g)(f', g') = (ff', gg')$. There is a functor $P_1 : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A}$ defined by $P_1(A, B) = A$ and $P_1(f, g) = f$, and an obvious P_2 . We will need the following.

Theorem 2.1. [15, p. 148] *If \mathfrak{A} is cocomplete, co-wellpowered and with a generating set, then every cocontinuous functor from \mathfrak{A} to a locally small category has a right adjoint.*

Proposition 2.2. *Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, $CoMon(\mathcal{C})$ be the category of comonoids of \mathcal{C} and $U : CoMon(\mathcal{C}) \rightarrow \mathcal{C}$ be the forgetful functor.*

- (i) *If \mathcal{C} is cocomplete, then $CoMon(\mathcal{C})$ is cocomplete and U preserves colimits.*
- (ii) *If furthermore \mathcal{C} is co-wellpowered, then so is $CoMon(\mathcal{C})$.*

For the fundamental concepts of category theory, we refer to [18], [20], [7], [24], [23], or [22].

Following [25], a **quiver** $Q = (Q_0, Q_1, s, t)$ consists of

- Q_0 a set of vertices,
- Q_1 a set of arrows,

- $s : Q_1 \rightarrow Q_0$ a map from arrows to vertices, mapping an arrow to its starting point,
- $t : Q_1 \rightarrow Q_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

We will represent an element $\alpha \in Q_1$ by drawing an arrow from its starting point $s(\alpha)$ to its endpoint $t(\alpha)$ as follows: $s(\alpha) \xrightarrow{\alpha} t(\alpha)$.

A **representation** $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of a quiver Q is a collection of k -vector spaces M_i one for each vertex $i \in Q_0$, and a collection of k -linear maps $\varphi_\alpha : M_{s(\alpha)} \rightarrow M_{t(\alpha)}$ one for each arrow $\alpha \in Q_1$.

A representation M is called **finite-dimensional** if each vector space M_i is finite-dimensional.

Let Q be a quiver and let $M = (M_i, \varphi_\alpha)$, $M' = (M'_i, \varphi'_\alpha)$ be two representations of Q . A **morphism** of representations $f : M \rightarrow M'$ is a collection $(f_i)_{i \in Q_0}$ of k -linear maps $f_i : M_i \rightarrow M'_i$ such that for each arrow $s(\alpha) \xrightarrow{\alpha} t(\alpha)$ in Q_1 the diagram

$$(2.1) \quad \begin{array}{ccc} M_{s(\alpha)} & \xrightarrow{\phi_\alpha} & M_{t(\alpha)} \\ f_{s(\alpha)} \downarrow & & \downarrow f_{t(\alpha)} \\ M'_{s(\alpha)} & \xrightarrow{\phi'_\alpha} & M'_{t(\alpha)} \end{array}$$

commutes.

A morphism of representations $f = (f_i) : M \rightarrow M'$ is an equivalence if each f_i is bijective. The class of all representations that are isomorphic to a given representation M is called the **isoclass** of M .

This gives rise to define a category $Rep_k(Q)$ of k -linear representations of Q . We denote by $rep_k(Q)$ the full subcategory of $Rep_k(Q)$ consisting of the finite dimensional representations.

Given two representations $M = (M_i, \phi_\alpha)$ and $M' = (M'_i, \phi'_\alpha)$ of Q , the representation

$$(2.2) \quad M \oplus M' = (M_i \oplus M'_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \phi'_\alpha \end{bmatrix})$$

is the **direct sum** of M and M' in $Rep_k(Q)$ [5, p. 71].

A nonzero representation of a quiver Q is said to be **indecomposable** if it is not isomorphic to a direct sum of two nonzero representations [14, p. 21].

Following [25, p. 114], the **path algebra** kQ of a quiver Q is the algebra with basis the set of all paths in the quiver Q and with multiplication defined on two basis elements c, c' by

$$c \cdot c' = \begin{cases} cc', & \text{if } s(c') = t(c) \\ 0, & \text{otherwise.} \end{cases}$$

We will need the following propositions.

Proposition 2.3. [5, p. 70] *Let Q be a finite quiver. Then $Rep_k(Q)$ and $rep_k(Q)$ are k -linear abelian categories.*

Proposition 2.4. [5, p. 74] *Let Q be a finite, connected, and acyclic quiver. There exists an equivalence of categories $Mod\,kQ \simeq Rep_k(Q)$ that restricts to an equivalence $mod\,kQ \simeq rep_k(Q)$, where kQ is the path algebra of Q , $Mod\,kQ$ denotes the category of right kQ -modules, and $mod\,kQ$ denotes the full subcategory of $Mod\,kQ$ consisting of the finitely generated right kQ -modules.*

We will also need the following theorem.

Theorem 2.5. [15, p. 148] *If \mathfrak{A} is cocomplete, co-wellpowered and with a generating set, then every cocontinuous functor from \mathfrak{A} to a locally small category has a right adjoint.*

This is a very brief review of the basic concepts involved with our work. For the basic notions of quiver representations theory, we refer the reader to [5], [25], [6], [9], [14], [10], [30].

3. n -representations of Quivers: Basic Concepts

Let $Q = (Q_0, Q_1, s, t)$, $Q' = (Q'_0, Q'_1, s', t')$ be quivers.

Definition 3.1. [3] A **2-representation** of (Q, Q') (or a **birepresentation** of (Q, Q')) is a triple $\bar{M} = ((M_i, \phi_\alpha), (M'_{i'}, \phi'_{\beta}), (\psi_\beta^\alpha))_{i \in Q_0, i' \in Q'_0, \alpha \in Q_1, \beta \in Q'_1}$, where $(M_i, \phi_\alpha), (M'_{i'}, \phi'_{\beta})$ are representations of Q, Q' respectively, and (ψ_β^α) is a collection of k -linear maps $\psi_\beta^\alpha : M_{t(\alpha)} \rightarrow M'_{s(\beta)}$, one for each pair of arrows $(\alpha, \beta) \in Q_1 \times Q'_1$.

Unless confusion is possible, we denote a birepresentation simply by $\bar{M} = (M, M', \psi)$. Next, we inductively define n -representations for any integer $n \geq 2$.

For any $m \in \{2, \dots, n\}$, let $Q_m = (Q_0^{(m)}, Q_1^{(m)}, s^{(m)}, t^{(m)})$ be a quiver. A **n -representation** of (Q_1, Q_2, \dots, Q_n) is $(2n-1)$ -tuple $\bar{V} = (V^{(1)}, V^{(2)}, \dots, V^{(n)}, \psi_1, \psi_2, \dots, \psi_{n-1})$, where for every $m \in \{1, 2, \dots, n\}$, $V^{(m)}$ is a representation of Q_m , and $(\psi_{\gamma^{(m-1)}}^{\gamma^{(m)}})$ is a collection of k -linear maps

$$\psi_{m\gamma^{(m-1)}}^{\gamma^{(m)}} : V_{t^{(m-1)}(\gamma^{(m-1)})}^{(m-1)} \rightarrow V_{s^{(m)}(\gamma^{(m)})}^{(m-1)},$$

one for each pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in Q_1^{(m-1)} \times Q_1^{(m)}$ and $m \in \{2, \dots, n\}$.

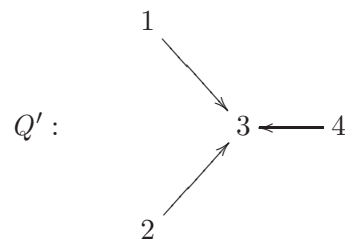
Remark 3.2. [3]

- (i) When no confusion is possible, we simply write s, t instead of s', t' respectively, and for every $m \in \{1, 2, \dots, n\}$, we write s, t instead of $s^{(m)}, t^{(m)}$ respectively.
- (ii) It is clear that if $(V^{(1)}, V^{(2)}, \dots, V^{(n)}, \psi_1, \psi_2, \dots, \psi_{n-1})$ is an n -representation of (Q_1, Q_2, \dots, Q_n) , then $(V^{(1)}, V^{(2)}, \dots, V^{(n-1)}, \psi_1, \psi_2, \dots, \psi_{n-2})$ is an $(n-1)$ -representation of $(Q_1, Q_2, \dots, Q_{n-1})$ for every integer $n \geq 2$.
- (iii) Part (ii) implies that for any integer $n > 2$, n -representations roughly inherit all the properties and the universal constructions that $(n-1)$ -representations have. Thus, we mostly focus on studying birepresentations since they can be regarded as a mirror in which one can see a clear decription of n -representations for any integer $n > 2$.

Example 3.3. [3] Let Q, Q' be the following quivers

(3.1)

$$Q : \quad 1 \longrightarrow 2$$



and consider the following:

$$(3.2) \quad \begin{array}{ccc} & \begin{array}{c} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \swarrow \\ k \end{array} & \begin{array}{c} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \leftarrow \\ k \end{array} \\ M & k^2 & M' \\ & \begin{array}{c} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \nearrow \\ k \end{array} & \begin{array}{c} k \xleftarrow{1} k \end{array} \end{array}$$

Then M (respectively M') is a representation of Q ((respectively Q') [25]. The following are birepresentations of (Q, Q') .

$$(3.3) \quad \begin{array}{ccc} & \begin{array}{c} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \swarrow \\ k \end{array} & \begin{array}{c} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \leftarrow \\ k \end{array} \\ \bar{M} & k^2 & k \xleftarrow{1} k \\ & \begin{array}{c} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \nearrow \\ k \end{array} & \end{array}$$

1

$$(3.4) \quad \begin{array}{ccc} & \begin{array}{c} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \swarrow \\ k \end{array} & \begin{array}{c} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \leftarrow \\ k \end{array} \\ \bar{N} & k^2 & k \xleftarrow{1} k \\ & \begin{array}{c} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \nearrow \\ k \end{array} & \end{array}$$

0

Definition 3.4. [3] Let $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$ be birepresentations of (Q, Q') . Write $V = (V_i, \phi_\alpha)$, $V' = (V'_i, \mu_\beta)$, $W = (W_i, \phi'_\alpha)$, $W' = (W'_i, \mu'_\beta)$. A morphism of birepresentations $\bar{f} : \bar{V} \rightarrow \bar{W}$ is a pair $\bar{f} = (f, f')$, where $f = (f_i) : (V_i, \phi_\alpha) \rightarrow (W_i, \phi'_\alpha)$, $f' = (f'_i) : (V'_i, \mu_\beta) \rightarrow (W'_i, \mu'_\beta)$

are morphisms in $\text{Rep}_k(Q)$, $\text{Rep}_k(Q')$ respectively such that the following diagram commutes.

$$(3.5) \quad \begin{array}{ccccc} V_{s(\alpha)} & \xrightarrow{\phi_\alpha} & V_{t(\alpha)} & & \\ & \searrow f_{s(\alpha)} & \downarrow \psi_\beta^\alpha & \searrow f_{t(\alpha)} & \\ & W_{s(\alpha)} & \xrightarrow{\phi'_\alpha} & W_{t(\alpha)} & \\ & & \downarrow \psi_{s(\beta)}' & \downarrow \psi_{t(\beta)}' & \\ & & V_{s(\beta)}' & \xrightarrow{\mu_\beta} & V_{t(\beta)}' \\ & & \searrow f_{s(\beta)}' & & \searrow f_{t(\beta)}' \\ & & W_{s(\beta)}' & \xrightarrow{\mu'_\beta} & W_{t(\beta)}' \end{array}$$

The composition of two maps (f, f') and (g, g') can be depicted as the following diagram.

$$(3.6) \quad \begin{array}{ccccccc} V_{s(\alpha)} & \xrightarrow{\phi_\alpha} & V_{t(\alpha)} & & & & \\ & \searrow f_{s(\alpha)} & \downarrow \psi_\beta^\alpha & \searrow f_{t(\alpha)} & & & \\ & W_{s(\alpha)} & \xrightarrow{\phi'_\alpha} & W_{t(\alpha)} & & & \\ & & \searrow g_{s(\alpha)} & \downarrow \psi_{s(\beta)}' & \searrow g_{t(\alpha)} & & \\ & & U_{s(\alpha)} & \xrightarrow{\phi''_\alpha} & U_{t(\alpha)} & & \\ & & & \downarrow \psi_{t(\beta)}' & & & \\ & & & V_{s(\beta)}' & \xrightarrow{\mu_\beta} & V_{t(\beta)}' & \\ & & & \searrow f_{s(\beta)}' & & \searrow f_{t(\beta)}' & \\ & & & W_{s(\beta)}' & \xrightarrow{\mu'_\beta} & W_{t(\beta)}' & \\ & & & & \downarrow \psi_{s(\beta)}' & \downarrow \psi_{t(\beta)}' & \\ & & & & U_{s(\beta)}' & \xrightarrow{\mu''_\beta} & U_{t(\beta)}' \end{array}$$

In general, if $\bar{V} = (V^{(1)}, V^{(2)}, \dots, V^{(n)}, \psi_1, \psi_2, \dots, \psi_{n-1})$, $\bar{W} = (W^{(1)}, W^{(2)}, \dots, W^{(n)}, \psi'_1, \psi'_2, \dots, \psi'_{n-1})$ are n -representations of (Q_1, Q_2, \dots, Q_n) , then a morphism of n -representations $f : \bar{V} \rightarrow \bar{W}$ is n -tuple $\bar{f} = (f^{(1)}, f^{(2)}, \dots, f^{(n-1)})$, where

$$f^{(m)} = (f_{i(m)}^{(m)}) : (V_{i(m)}, \phi_{\gamma(m)}^{i(m)}) \rightarrow (W_{i(m)}, \mu_{\gamma(m)}^{i(m)}),$$

is a morphism in $\text{Rep}_k(Q_m)$ for any $m \in \{2, \dots, n\}$, and for each pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in Q_1^{(m-1)} \times Q_1^{(m)}$ the following diagram is commutative.

for every $m \in \{2, \dots, n\}$.

A morphism of n -representations can be depicted as:

Remark 3.5. [3]

- (i) The above definition gives rise to form a category $Rep_{(Q,Q')}$ of k -linear birepresentations of (Q, Q') . We denote by $rep_{(Q,Q')}$ the full subcategory of $Rep_{(Q,Q')}$ consisting of the finite dimensional birepresentations. Similarly, it also creates a category $Rep_{(Q_1, Q_2, \dots, Q_n)}$ of n -representations. We denote $rep_{(Q_1, Q_2, \dots, Q_n)}$ the full subcategory of $Rep_{(Q_1, Q_2, \dots, Q_n)}$ consisting of the finite dimensional n -representations.
- (ii) For any $m \in \{2, \dots, n\}$, let $Q_m = (Q_0^{(m)}, Q_1^{(m)}, s^{(m)}, t^{(m)})$ be a quiver and fix $j \in \{2, \dots, n\}$. Let $\Upsilon_{Rep_k(Q_j)}$ be the subcategory of $Rep_{(Q_1, Q_2, \dots, Q_n)}$ whose objects are $(2n-1)$ -tuples $\vec{X} = (0, 0, \dots, V^{(j)}, 0, \dots, 0, \psi_1, \psi_2, \dots, \psi_{n-1})$, where $V^{(j)}$ is a representation of Q_j , and $\psi^{\gamma^{(m)}}_{m\gamma^{(m-1)}} = 0$ for every pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in Q_1^{(m-1)} \times Q_1^{(m)}$ and $m \in \{2, \dots, n\}$. Then $\Upsilon_{Rep_k(Q_j)}$ is clearly a full subcategory of $Rep_{(Q_1, Q_2, \dots, Q_n)}$. Notably, we have an equivalence of categories $\Upsilon_{Rep_k(Q_j)} \simeq Rep_k(Q_j)$, and thus by Proposition (2.4), we have $Rep_k(Q_j) \simeq \Upsilon_{Rep_k(Q_j)} \simeq Mod k Q_j$. It turns out that the category $Rep_k(Q_j)$ and $Mod k Q_j$ can be identified as full subcategories of $Rep_{(Q_1, Q_2, \dots, Q_n)}$.

The category $\Upsilon_{Rep_k(Q_j)}$ has a full subcategory $\tilde{\Upsilon}_{rep_k(Q_j)}$ when we restrict the objects on the finite dimensional representations. Therefore, we also have $rep_k(Q_j) \simeq \Upsilon_{rep_k(Q_j)} \simeq mod\ kQ_j$.

Remark 3.6. Let \mathcal{B}_0 be the class of all quivers. One might consider the class \mathcal{B}_0 and full subcategories of the categories of birepresentations of quivers to build a bicategory. Indeed, there is a bicategory \mathcal{B} consists of

- the objects or the 0-cells of \mathcal{B} are simply the elements of \mathcal{B}_0

- for each $Q, Q' \in \mathcal{B}_0$, we have $\mathcal{B}(Q, Q') = \text{Rep}_k(Q) \times \text{Rep}_k(Q')$, whose objects are the 1-cells of \mathcal{B} , and whose morphisms are the 2-cells of \mathcal{B}
- for each $Q, Q', Q'' \in \mathcal{B}_0$, a composition functor

$$\mathcal{F} : \text{Rep}_k(Q') \times \text{Rep}_k(Q'') \times \text{Rep}_k(Q) \times \text{Rep}_k(Q') \rightarrow \text{Rep}_k(Q) \times \text{Rep}_k(Q'')$$

defined by:

$$\begin{aligned} \mathcal{F}((N', N''), (M, M')) &= (M, N''), \quad \mathcal{F}((g', g''), (f, f')) = (f, g'') \text{ on 1-cells} \\ &\quad (M, M'), (N', N'') \text{ and 2-cells } (f, f'), (g', g''). \end{aligned}$$

- for any $Q \in \mathcal{B}_0$ and for each $(M, M') \in \mathcal{B}(Q, Q)$, we have $\mathcal{F}((M, M'), (M, M)) = (M, M')$ and $\mathcal{F}((M', M'), (M, M')) = (M, M')$. Furthermore, for any 2-cell (f, f') , we have $\mathcal{F}((f', f'), (f, f')) = (f, f')$ and $\mathcal{F}((f, f'), (f, f)) = (f, f')$. Thus, the identity and the unit coherence axioms hold.

The rest of bicategories axioms are obviously satisfied. For each $Q, Q' \in \mathcal{B}_0$, let $\mathcal{B}_{(Q, Q')}$ be the full subcategory of $\text{Rep}_{(Q, Q')}$ whose objects are the triples (X, X', Ψ) , where $(X, X') \in \text{Rep}_k(Q) \times \text{Rep}_k(Q')$ and $\Psi_\beta^\alpha = 0$ for every pair of arrows $(\alpha, \beta) \in \mathcal{Q}_1 \times \mathcal{Q}'_1$, and whose morphisms are usual morphisms of birepresentations between them. Clearly, $\mathcal{B}_{(Q, Q')} \cong \text{Rep}_k(Q) \times \text{Rep}_k(Q')$ for any $Q, Q' \in \mathcal{B}_0$. Thus, by considering the class \mathcal{B}_0 and the full subcategories described above of the birepresentations categories of quivers, we can always build a bicategory as above.

Obviously, the discussion above implies that for each $Q, Q' \in \mathcal{B}_0$, the product category $\text{Rep}_k(Q) \times \text{Rep}_k(Q')$ can be viewed as a full subcategory of $\text{Rep}_{(Q, Q')}$. Further, it implies that the product category $\text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \times \dots \times \text{Rep}_k(Q_n)$ can be viewed as a full subcategory of $\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$, where $Q_1, Q_2, \dots, Q_n \in \mathcal{B}_0$ and $n \geq 2$.

We also have the same analogue if we replace $\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$, by $\text{rep}_{(Q_1, Q_2, \dots, Q_n)}$, and $\text{Rep}_k(Q_1)$, $\text{Rep}_k(Q_2)$, \dots , $\text{Rep}_k(Q_n)$ by $\text{rep}_k(Q_1)$, $\text{rep}_k(Q_2)$, \dots , $\text{rep}_k(Q_n)$ respectively. For the basic notions of bicategories, we refer the reader to [19].

Definition 3.7. [3] Let $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$ be birepresentations of (Q, Q') . Write $V = (V_i, \phi_\alpha)$, $V' = (V'_{i'}, \phi'_\beta)$, $W = (W_i, \mu_\alpha)$, $W' = (W'_{i'}, \mu'_\beta)$. Then

$$(3.9) \quad \bar{V} \oplus \bar{W} = ((V_i \oplus W_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \mu_\alpha \end{bmatrix}), (V'_{i'} \oplus W'_{i'}, \begin{bmatrix} \phi'_\alpha & 0 \\ 0 & \mu'_\alpha \end{bmatrix}), \begin{bmatrix} \psi_\beta^\alpha & 0 \\ 0 & \psi'_\beta \end{bmatrix}),$$

where $(V_i \oplus W_i, \begin{bmatrix} \phi_\alpha & 0 \\ 0 & \mu_\alpha \end{bmatrix})$, $(V'_{i'} \oplus W'_{i'}, \begin{bmatrix} \phi'_\alpha & 0 \\ 0 & \mu'_\alpha \end{bmatrix})$ are the direct sums of (V_i, ϕ_α) , (W_i, μ_α) and $(V'_{i'}, \phi'_\beta)$, $(W'_{i'}, \mu'_\beta)$ in $\text{Rep}_k(Q)$, $\text{Rep}_k(Q')$ respectively, is a birepresentation of (Q, Q') called the **direct sum** of \bar{V} , \bar{W} (in $\text{Rep}_{(Q, Q')}$).

Similarly, direct sums in $\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$ can be defined.

Definition 3.8. [3] A birepresentation $\bar{V} \in \text{Rep}_{(Q, Q')}$ is called **indecomposable** if $\bar{M} \neq 0$ and \bar{M} cannot be written as a direct sum of two nonzero birepresentations, that is, whenever $\bar{M} \cong \bar{L} \oplus \bar{N}$ with $\bar{L}, \bar{N} \in \text{Rep}_{(Q, Q')}$, then $\bar{L} = 0$ or $\bar{N} = 0$.

We end the section with the following consequence.

Theorem 3.9. [3] *The category $\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$ is a k -linear abelian category for any integer $n \geq 2$.*

For more concrete examples and proofs of the theorems above, we refer the reader to [3].

4. Completeness and Cocompleteness in The Categories of n -representations

Recall that a category \mathfrak{C} is cocomplete when every functor $\mathfrak{F} : \mathfrak{D} \rightarrow \mathfrak{C}$, with \mathfrak{D} a small category has a colimit [11]. For the basic notions of cocomplete categories and examples, we refer to [4], [11], or [26]. A functor is *cocontinuous* if it preserves all small colimits [15, p. 142].

Proposition 4.1. *The category $\text{Rep}_{(Q, Q')}$ is complete and the forgetful functor*

$$\mathcal{U} : \text{Rep}_{(Q, Q')} \rightarrow \text{Rep}_k(Q) \times \text{Rep}_k(Q')$$

is continuous. Furthermore, the limit of objects in $\text{Rep}_{(Q, Q')}$ can be obtained by the corresponding construction for objects in $\text{Rep}_k(Q) \times \text{Rep}_k(Q')$.

Proof.

Let \mathcal{D} be a small category, and let $\mathcal{F} : \mathcal{D} \rightarrow \text{Rep}_{(Q, Q')}$ be a functor, and consider the composition of the following functors.

$$(4.1) \quad \mathcal{D} \xrightarrow{\mathcal{F}} \text{Rep}_{(Q, Q')} \xrightarrow{\mathcal{U}} \text{Rep}_k(Q) \times \text{Rep}_k(Q') \xrightarrow{\mathcal{P}_1} \text{Rep}_k(Q)$$

$$(4.2) \quad \mathcal{D} \xrightarrow{\mathcal{F}} \text{Rep}_{(Q, Q')} \xrightarrow{\mathcal{U}'} \text{Rep}_k(Q) \times \text{Rep}_k(Q') \xrightarrow{\mathcal{P}_2} \text{Rep}_k(Q')$$

where $\mathcal{U}, \mathcal{U}'$ are the obvious forgetful functors, and $\mathcal{P}_1, \mathcal{P}_2$ are the projection functors.

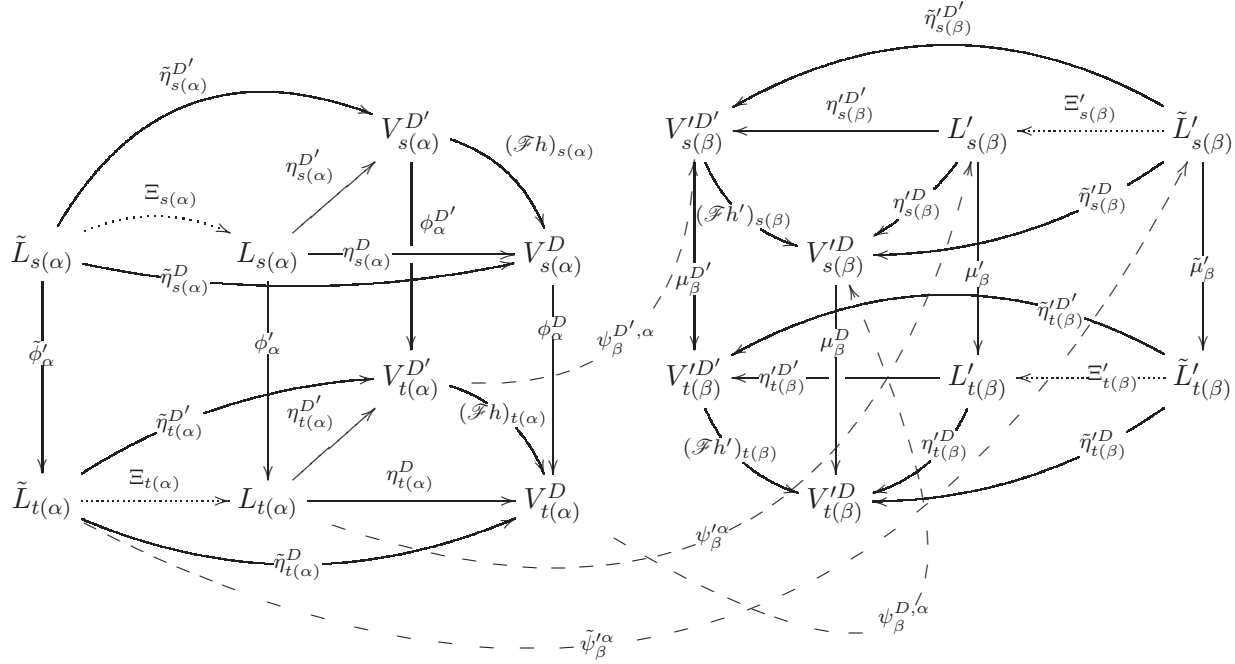
For all $D \in \mathcal{D}$, let $\mathcal{F}D = \bar{V} = (V, V', \psi^D)$. Since $\text{Rep}_k(Q), \text{Rep}_k(Q')$ are complete categories, the functors $\mathcal{P}_1\mathcal{U}\mathcal{F}, \mathcal{P}_2\mathcal{U}'\mathcal{F}$ have limits. Let $(L, (\eta^D)_{D \in \mathcal{D}}), (L', (\eta'^D)_{D \in \mathcal{D}})$ be limits of $\mathcal{P}_1\mathcal{U}\mathcal{F}, \mathcal{P}_2\mathcal{U}'\mathcal{F}$ respectively, where η^D, η'^D are the morphisms

$$(4.3) \quad L \xrightarrow{\eta^D} V^D, \quad L' \xrightarrow{\eta'^D} V'^D,$$

for every $D \in \mathcal{D}$. Write $L = (L_i, \phi'_\alpha)$, $L' = (L'_i, \mu'_\beta)$, $V^D = (V_i^D, \phi_\alpha^D)$, $V^{D'} = (V_i^{D'}, \phi_\alpha^{D'})$, $V'^{D'} = (V'_i^{D'}, \mu_\beta^{D'})$, $V'^D = (V'_i^D, \mu_\beta^D)$.

Let $h : D \rightarrow D'$ be a morphism in \mathcal{D} and consider the following diagram

(4.4)



Clearly, $(L'_{s(\beta)}, (\eta_{s(\beta)}^D)_{D \in \mathcal{D}})$ can be viewed as a limit of a functor $\rho : \mathcal{D} \rightarrow \text{Vect}_k$. Furthermore, we have for any $D, D' \in \mathcal{D}$,

$$\begin{aligned}
 (\mathcal{F}h')_{s(\beta)} \psi_{\beta}^{D', \alpha} \eta_{t(\alpha)}^{D'} &= \psi_{\beta}^{D, \alpha} (\mathcal{F}h)_{t(\alpha)} \eta_{t(\alpha)}^{D'} \\
 &\quad (\text{since } \mathcal{F}h = (\mathcal{F}h, \mathcal{F}h') \text{ is a morphism in } \text{Rep}_{(Q, Q')}) \\
 &= \psi_{\beta}^{D, \alpha} \eta_{t(\alpha)}^D \\
 &\quad (\text{since } (L, (\eta^D)_{D \in \mathcal{D}}) \text{ is a limit of } \mathcal{P}_1 \mathcal{U} \mathcal{F})
 \end{aligned}$$

Thus, $(L_{t(\alpha)}, (\psi_{t(\alpha)}^D, \eta_{t(\alpha)}^D)_{D \in \mathcal{D}})$ is a cone on ρ . Since Vect_k is complete, there exists a unique k -linear map $\psi_{\beta}^{\prime \alpha} : L_{t(\alpha)} \rightarrow L'_{s(\beta)}$ with $\eta_{s(\beta)}^D \psi_{\beta}^{\prime \alpha} = \psi_{\beta}^{D, \alpha} \eta_{t(\alpha)}^D$ for every $D \in \mathcal{D}$ and for each pair of arrows $(\alpha, \beta) \in Q_1 \times Q'_1$. Hence, $\bar{\eta}^D$ is a morphism in $\text{Rep}_{(Q, Q')}$ for any $D \in \mathcal{D}$. Let $\bar{L} = (L, L', \psi')$, $\bar{\eta}^D = (\eta^D, \eta'^D)$. We claim that $(\bar{L}, (\bar{\eta}^D)_{D \in \mathcal{D}})$ is a limit of \mathcal{F} . Then obviously all we need is to show that for any cone $(\tilde{L}, (\tilde{\eta}^D)_{D \in \mathcal{D}})$, there exists a unique morphism $\bar{\Xi} = (\Xi, \Xi')$ in $\text{Rep}_{(Q, Q')}$ with $\bar{\eta}^D \bar{\Xi} = \tilde{\eta}^D$ for every $D \in \mathcal{D}$. Let $(\tilde{L}, (\tilde{\eta}^D)_{D \in \mathcal{D}})$ be a cone on \mathcal{F} and write $\tilde{L} = (\tilde{L}, \tilde{L}', \tilde{\psi}')$. Since $\text{Rep}_k(Q)$, $\text{Rep}_k(Q')$ are complete categories, exists unique morphisms Ξ, Ξ' in $\text{Rep}_k(Q)$, $\text{Rep}_k(Q')$ respectively such that $\eta^D \Xi = \tilde{\eta}^D$, $\eta'^D \Xi' = \tilde{\eta}'^D$ for every $D \in \mathcal{D}$. It remains to show that $\bar{\Xi} = (\Xi, \Xi')$ is a morphism in $\text{Rep}_{(Q, Q')}$.

For any $D \in \mathcal{D}$, we have

$$\begin{aligned}
\eta'_{s(\beta)} \Xi'_{s(\beta)} \tilde{\psi}'_{\beta} &= \tilde{\eta}'_{s(\beta)} \tilde{\psi}'_{\beta} \\
&\quad (\text{since } \eta'^D \Xi' = \tilde{\eta}'^D) \\
&= \psi_{\beta}^{D,\alpha} \tilde{\eta}_{t(\alpha)}^D \\
&\quad (\text{since } \tilde{\eta}'^D \text{ is a morphism in } Rep_{(Q,Q')}) \\
&= \psi_{\beta}^{D,\alpha} \eta_{t(\alpha)}^D \Xi_{t(\alpha)} \\
&\quad (\text{since } \eta'^D \Xi = \tilde{\eta}'^D) \\
&= \eta'_{s(\beta)} \psi_{\beta}^{\prime\alpha} \Xi_{t(\alpha)} \\
&\quad (\text{since } \tilde{\eta}'^D \text{ is a morphism in } Rep_{(Q,Q')})
\end{aligned}$$

Notably, $(\tilde{L}_{t(\alpha)}, (\tilde{\eta}'_{s(\beta)} \tilde{\psi}'_{\beta})_{D \in \mathcal{D}})$ can be viewed as a limit of a functor $\rho' : \mathcal{D} \rightarrow Vect_k$. Furthermore, $(\tilde{L}_{t(\alpha)}, (\tilde{\eta}'_{s(\beta)} \tilde{\psi}'_{\beta})_{D \in \mathcal{D}})$ is a cone on ρ' since

$$\begin{aligned}
(\mathcal{F}h')_{s(\beta)} \tilde{\eta}'_{s(\beta)} \tilde{\psi}'_{\beta} &= \mathcal{F}h'_{s(\beta)} \eta'_{s(\beta)} \Xi'_{s(\beta)} \tilde{\psi}'_{\beta} \\
&\quad (\text{since } \tilde{\eta}'^{D'} = \tilde{\eta}^{D'} \Xi) \\
&= \eta'_{s(\beta)} \Xi'_{s(\beta)} \tilde{\psi}'_{\beta} \\
&\quad (\text{since } \mathcal{F}h \tilde{\eta}^{D'} = \tilde{\eta}^D) \\
&= \tilde{\eta}'_{s(\beta)} \tilde{\psi}'_{\beta} .
\end{aligned}$$

Since $Vect_k$ is complete, it follows from the universal property of the limit that $\Xi'_{s(\beta)} \tilde{\psi}'_{\beta} = \psi_{\beta}^{\prime\alpha} \Xi_{t(\alpha)}$. Thus, Ξ is a morphism in $Rep_{(Q,Q')}$. Thus, $(\bar{L}, (\bar{\eta}^D)_{D \in \mathcal{D}})$ is a limit of \mathcal{F} , as desired. \square

From Proposition (4.1) and Remark (3.2), we obtain.

Proposition 4.2. *The category $Rep_{(Q_1, Q_2, \dots, Q_n)}$ is complete and the forgetful functor*

$$\mathcal{U}_n : Rep_{(Q_1, Q_2, \dots, Q_n)} \rightarrow Rep_k(Q_1) \times Rep_k(Q_2) \times \dots \times Rep_k(Q_n)$$

is continuous. Furthermore, the limit of objects in $Rep_{(Q_1, Q_2, \dots, Q_n)}$ can be obtained by the corresponding construction for objects in $Rep_k(Q_1) \times Rep_k(Q_2) \times \dots \times Rep_k(Q_n)$.

Proof. \square

Proposition 4.3. *The category $Rep_{(Q, Q')}$ is cocomplete and the forgetful functor*

$$\mathcal{U} : Rep_{(Q, Q')} \rightarrow Rep_k(Q) \times Rep_k(Q')$$

is cocontinuous. Moreover, the colimit of objects in $Rep_{(Q, Q')}$ can be obtained by the corresponding construction for objects in $Rep_k(Q) \times Rep_k(Q')$.

Proof.

Let \mathcal{D} be a small category, and let $\mathcal{F} : \mathcal{D} \rightarrow Rep_{(Q, Q')}$ be a functor, and consider the composition of the following functors.

$$(4.5) \quad \mathcal{D} \xrightarrow{\mathcal{F}} Rep_{(Q, Q')} \xrightarrow{\mathcal{U}} Rep_k(Q) \times Rep_k(Q') \xrightarrow{\mathcal{P}_1} Rep_k(Q)$$

$$\begin{aligned} \zeta_{s(\beta)}^{D', \alpha} \psi_{\beta}^{D', \alpha} (\mathcal{F}h')_{t(\alpha)} &= \zeta_{s(\beta)}^{D'} (\mathcal{F}h)_{s(\beta)} \psi_{\beta}^{D, \alpha} \\ &\quad (\text{since } \mathcal{F}h = (\mathcal{F}h, \mathcal{F}h') \text{ is a morphism in } Rep_{(Q, Q')}) \\ &= \zeta_{s(\beta)}^{D'} \psi_{\beta}^{D, \alpha} \\ &\quad (\text{since } (C', (\zeta^D)_{D \in \mathcal{D}}) \text{ is a colimit of } \mathcal{P}_2\mathcal{UF}) \end{aligned}$$

Therefore, $(C'_{s(\beta)}, (\zeta_{s(\beta)}^D, \psi^D)_{D \in \mathcal{D}})$ is a cocone on v . Since Vec_k is cocomplete, there exists a unique k -linear map $\psi'_\beta : C_{t(\alpha)} \rightarrow C'_{s(\beta)}$ with $\psi'_\beta \zeta_{t(\alpha)}^D = \zeta_{s(\beta)}^D \psi_\beta^{D,\alpha}$ for every $D \in \mathcal{D}$ and for each pair of arrows $(\alpha, \beta) \in Q_1 \times Q'_1$.

It turns out that $\bar{\zeta}^D$ is a morphism in $Rep_{(Q,Q')}$ for any $D \in \mathcal{D}$. Let $\bar{C} = (C, C', \psi')$, $\bar{\zeta}^D = (\zeta^D, \zeta'^D)$. We claim that $(\bar{C}, (\bar{\zeta}^D)_{D \in \mathcal{D}})$ is a colimit of \mathcal{F} . To substantiate this claim, we need to show that for any cocone $(\tilde{C}, (\tilde{\zeta}^D)_{D \in \mathcal{D}})$, there exists a unique morphism $\bar{\Lambda} = (\Lambda, \Lambda')$ in $Rep_{(Q,Q')}$ with $\bar{\Lambda} \bar{\zeta}^D = \tilde{\zeta}^D$ for every $D \in \mathcal{D}$. Let $(\tilde{C}, (\tilde{\zeta}^D)_{D \in \mathcal{D}})$ be a cocone on \mathcal{F} and write $\tilde{C} = (\tilde{C}, \tilde{C}', \tilde{\psi}')$. Since $Rep_k(Q)$, $Rep_k(Q')$ are cocomplete categories, exists unique morphisms Λ , Λ' in $Rep_k(Q)$, $Rep_k(Q')$ respectively such that $\Lambda \zeta^D = \tilde{\zeta}^D$, $\Lambda' \zeta'^D = \tilde{\zeta}'^D$ for every $D \in \mathcal{D}$. It remains to show that $\bar{\Lambda} = (\Lambda, \Lambda')$ is a morphism in $Rep_{(Q,Q')}$. For any $D \in \mathcal{D}$, we have

$$\begin{aligned} \tilde{\psi}'_\beta \Lambda_{t(\alpha)} \zeta_{t(\alpha)}^D &= \tilde{\psi}'_\beta \tilde{\zeta}_{t(\alpha)}^D \\ &\quad (\text{since } \bar{\Lambda} \bar{\zeta}^D = \tilde{\zeta}^D) \\ &= \tilde{\zeta}_{s(\beta)}^D \psi_\beta^{D,\alpha} \\ &\quad (\text{since } \tilde{\zeta}^D \text{ is a morphism in } Rep_{(Q,Q')}) \\ &= \Lambda'_{s(\beta)} \zeta_{s(\beta)}^D \psi_\beta^{D,\alpha} \\ &\quad (\text{since } \bar{\Lambda} \bar{\zeta}^D = \tilde{\zeta}^D) \\ &= \Lambda'_{s(\beta)} \psi'_\beta \zeta_{t(\alpha)}^D \\ &\quad (\text{since } \zeta^D \text{ is a morphism in } Rep_{(Q,Q')}) \end{aligned}$$

Notably, $(C_{t(\alpha)}, (\zeta_{t(\alpha)}^D)_{D \in \mathcal{D}})$ can be viewed as a colimit of a functor $v' : \mathcal{D} \rightarrow Vect_k$. Furthermore, $(\tilde{C}'_{s(\beta)}, (\tilde{\psi}'_\beta \tilde{\zeta}_{t(\alpha)}^D)_{D \in \mathcal{D}})$ is a cocone on v' since for any $D, D' \in \mathcal{D}$, we have

$$\begin{aligned} \tilde{\psi}'_\beta \tilde{\zeta}_{t(\alpha)}^{D'} (\mathcal{F}h)_{t(\alpha)} &= \tilde{\psi}'_\beta \Lambda_{t(\alpha)} \zeta_{t(\alpha)}^{D'} (\mathcal{F}h)_{t(\alpha)} \\ &\quad (\text{since } \bar{\Lambda} \bar{\zeta}^D = \tilde{\zeta}^D) \\ &= \tilde{\psi}'_\beta \Lambda_{t(\alpha)} \zeta_{t(\alpha)}^D \\ &\quad (\text{since } (C, (\zeta^D)_{D \in \mathcal{D}}) \text{ is a colimit of } \mathcal{P}_1 \mathcal{U} \mathcal{F}) \\ &= \tilde{\psi}'_\beta \tilde{\zeta}_{s(\beta)}^D \\ &\quad (\text{since } \bar{\Lambda} \bar{\zeta}^D = \tilde{\zeta}^D) \end{aligned}$$

Since Vec_k is cocomplete, it follows from the universal property of the colimit that $\Lambda'_{s(\beta)} \tilde{\psi}'_\beta = \tilde{\psi}'_\beta \Lambda_{t(\alpha)}$. Consequently, $\bar{\Lambda}$ is a morphism in $Rep_{(Q,Q')}$, which completes the proof. \square

From Proposition (4.3) and Remark (3.2), we obtain.

Proposition 4.4. *The category $Rep_{(Q_1, Q_2, \dots, Q_n)}$ is cocomplete and the forgetful functor*

$$\mathcal{U}_n : Rep_{(Q_1, Q_2, \dots, Q_n)} \rightarrow Rep_k(Q_1) \times Rep_k(Q_2) \times \dots \times Rep_k(Q_n)$$

is cocontinuous. Furthermore, the colimit of objects in $Rep_{(Q_1, Q_2, \dots, Q_n)}$ can be obtained by the corresponding construction for objects in $Rep_k(Q_1) \times Rep_k(Q_2) \times \dots \times Rep_k(Q_n)$.

Proof.

\square

5. n -quivers and n -representations

Let $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$ be birepresentations of (Q, Q') . For simplicity, we suppress k from the tensor product \otimes_k and use \otimes instead. Define

$$(5.1) \quad \bar{V} \otimes \bar{W} = (V \otimes W, V' \otimes W', \psi \otimes \psi') .$$

Then $\bar{V} \otimes \bar{W}$ is clearly a birepresentation of (Q, Q') . Let $\bar{f} : \bar{V} \rightarrow \bar{W}$, $\bar{g} : \bar{M} \rightarrow \bar{N}$ be morphisms in $Rep_{(Q, Q')}$ and write $\bar{V} = (V, V', \psi)$, $\bar{W} = (W, W', \psi')$, $\bar{M} = (M, M', \Psi)$, $\bar{N} = (N, N', \Psi')$. Define

$$(5.2) \quad \bar{f} \otimes \bar{g} : \bar{V} \otimes \bar{M} \rightarrow \bar{W} \otimes \bar{N} .$$

Then it is clear that $\bar{f} \otimes \bar{g}$ is a morphism in $Rep_{(Q, Q')}$, and hence the following diagram is commutative.

$$(5.3) \quad \begin{array}{ccccc} & & V_{s(\alpha)} \otimes M_{s(\alpha)} & \xrightarrow{\phi_\alpha \otimes \varphi_\alpha} & V_{t(\alpha)} \otimes M_{t(\alpha)} \\ & \swarrow f_{s(\alpha)} \otimes g_{s(\alpha)} & & \searrow f_{t(\alpha)} \otimes g_{t(\alpha)} & \downarrow \psi_\beta^\alpha \otimes \Psi_\beta^\alpha \\ W_{s(\alpha)} \otimes N_{s(\alpha)} & \xrightarrow{\phi'_\alpha \otimes \varphi'_\alpha} & W_{t(\alpha)} \otimes N_{t(\alpha)} & & \\ & \downarrow \psi_\beta^\alpha \otimes \Psi_\beta^\alpha & & & \\ & & V'_{s(\alpha)} \otimes M'_{s(\alpha)} & \xrightarrow{\mu_\beta \otimes \nu_\beta} & V'_{t(\alpha)} \otimes M'_{t(\alpha)} \\ & \swarrow f'_{s(\beta)} \otimes g'_{s(\beta)} & & \searrow f'_{t(\beta)} \otimes g'_{t(\beta)} & \\ W'_{s(\alpha)} \otimes N'_{s(\alpha)} & \xrightarrow{\mu'_\beta \otimes \nu'_\beta} & W'_{t(\alpha)} \otimes N'_{t(\alpha)} & & \end{array}$$

Thus, the category $Rep_{(Q, Q')}$ is a monoidal category, and hence by Remark (3.2), $Rep_{(Q_1, Q_2, \dots, Q_n)}$ is a monoidal category for any $n \geq 2$. For the basic notions of monoidal categories, we refer the reader to [13], [28], [8], and [12, Chapter 6].

Definition 5.1. Let $Q = (Q_0, Q_1, s, t)$, $Q' = (Q'_0, Q'_1, s', t')$ be quivers, and let $\{\varrho_\beta^\alpha : \alpha \in Q_1, \beta \in Q'_1\}$ be a collection of arrows $t(\alpha) \xrightarrow{\varrho_\beta^\alpha} s'(\beta)$ one for each pair of arrows $(\alpha, \beta) \in Q_1 \times Q'_1$.

A 2-*quiver induced by* (Q, Q') is a quiver $\mathcal{Q}_{(Q, Q')} = (\tilde{Q}_0, \tilde{Q}_1, s'', t'')$, where

- $\tilde{Q}_0 = Q_0 \sqcup Q'_0$,
- $\tilde{Q}_1 = Q_1 \sqcup Q'_1 \sqcup \{\varrho_\beta^\alpha : \alpha \in Q_1, \beta \in Q'_1\}$,
- $s'' : \tilde{Q}_1 \rightarrow \tilde{Q}_0$ a map from arrows to vertices, mapping an arrow to its starting point,
- $t'' : \tilde{Q}_1 \rightarrow \tilde{Q}_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

The notation \sqcup above denotes the disjoint union.

The above definition turns out that one can inductively define n -quivers for any integer $n \geq 2$. For any $m \in \{2, \dots, n\}$, let $Q_m = (Q_0^{(m)}, Q_1^{(m)}, s^{(m)}, t^{(m)})$ be a quiver, and let $\{\varrho_{m\gamma^{(m-1)}}^{\gamma^{(m)}} : \gamma^{(m)} \in$

$Q_1^{(m)}$, $m \in \{2, \dots, n\}$ be a collection of arrows

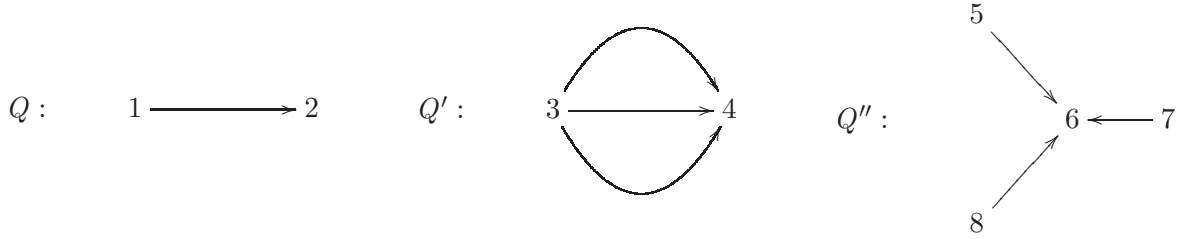
$$t^{(m-1)} \xrightarrow[\gamma^{(m-1)}]{\varrho^{(m)}} s^{(m)}$$

one for each pair of arrows $(\gamma^{(m-1)}, \gamma^{(m)}) \in Q_1^{(m-1)} \times Q_1^{(m)}$. An n -quiver induced by (Q_1, Q_2, \dots, Q_n) is a quiver $\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)} = (\hat{Q}_0, \hat{Q}_1, \hat{s}, \hat{t})$, where

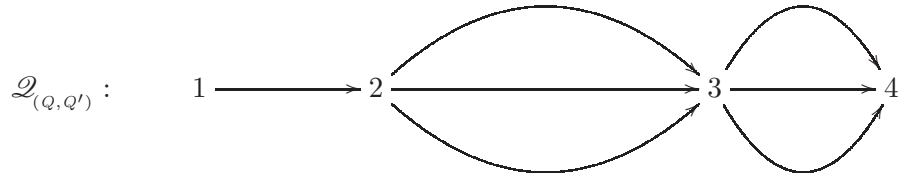
- $\hat{Q}_0 = \sqcup_{m=1}^n Q_0^{(m)}$,
- $\hat{Q}_1 = \sqcup_{m=1}^n Q_1^{(m)} \sqcup \{\varrho_m^{\gamma^{(m-1)}} : \gamma^{(m)} \in Q_1^{(m)}, m \in \{2, \dots, n\}\}$,
- $\hat{s} : \hat{Q}_1 \rightarrow \hat{Q}_0$ a map from arrows to vertices, mapping an arrow to its starting point,
- $\hat{t} : \hat{Q}_1 \rightarrow \hat{Q}_0$ a map from arrows to vertices, mapping an arrow to its terminal point.

Example 5.2.

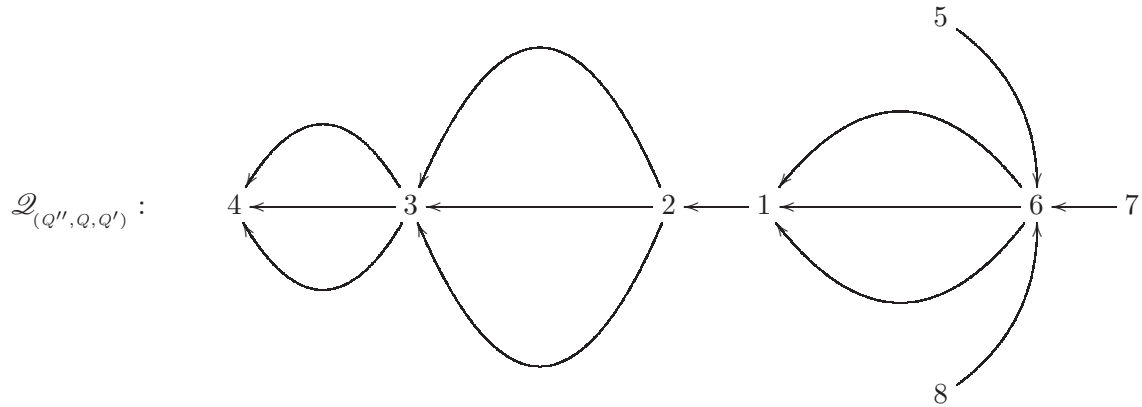
(5.4)



(5.5)



(5.6)



We have $k(Q) \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$, $k(Q') \cong \begin{bmatrix} k & k^3 \\ 0 & k \end{bmatrix}$, $k(\mathcal{Q}_{(Q,Q')}) \cong \begin{bmatrix} k & k & k^3 & k^9 \\ 0 & k & k^3 & k^9 \\ 0 & 0 & k & k^3 \\ 0 & 0 & 0 & k \end{bmatrix}$, and

$$k(\mathcal{Q}_{(Q'',Q,Q')}) \cong \begin{bmatrix} k & k & k^3 & k^9 & 0 & 0 & 0 & 0 \\ 0 & k & k^3 & k^9 & 0 & 0 & 0 & 0 \\ 0 & 0 & k & k^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 0 & 0 & 0 & 0 \\ k^3 & k^3 & k^9 & k^{27} & k & k & 0 & 0 \\ k^3 & k^3 & k^9 & k^{27} & 0 & k & 0 & 0 \\ k^3 & k^3 & k^9 & k^{27} & 0 & k & k & 0 \\ k^3 & k^3 & k^9 & k^{27} & 0 & k & 0 & k \end{bmatrix}.$$

Consider the following:

(5.7)

$$V : k \xrightarrow{1} k \quad V' : \begin{array}{ccc} & 1 & \\ \curvearrowright & & \curvearrowleft \\ k & \xrightarrow{1} & k \\ \curvearrowleft & & \curvearrowright \\ & 1 & \end{array} \quad V'' : \begin{array}{ccc} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \\ k \swarrow & & \searrow k \\ & k^2 & \\ k \nearrow & & \nwarrow k \\ & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \end{array}$$

Then V, V', V'' are clearly representations of Q, Q', Q'' respectively.
Now, consider the following:

(5.8)

$$\bar{V} : k \xrightarrow{1} k \begin{array}{ccc} & 1 & \\ \curvearrowright & & \curvearrowleft \\ & 0 & \\ \curvearrowleft & & \curvearrowright \\ & 1 & \end{array} \begin{array}{ccc} & 1 & \\ \curvearrowright & & \curvearrowleft \\ & 1 & \\ \curvearrowleft & & \curvearrowright \\ & 1 & \end{array} k$$

Remark 5.3.

$$k\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)} \cong \begin{bmatrix} k\mathcal{Q}_{(Q_1, Q_2, \dots, Q_{n-1})} & k\Omega_n \\ 0 & kQ_n \end{bmatrix} \quad \text{or} \quad k\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)} \cong \begin{bmatrix} k\mathcal{Q}_{(Q_1, Q_2, \dots, Q_{n-1})} & 0 \\ k\Omega_n & kQ_n \end{bmatrix}$$

(ii) We note that \bar{V} in the previous example can be identified as a birepresentation of (Q, Q') . Indeed, every representation of $\mathcal{Q}_{(Q, Q')}$ can be identified as a birepresentation of (Q, Q') . Conversely, every birepresentation of (Q, Q') can be viewed as a representation of $\mathcal{Q}_{(Q, Q')}$. Similarly, \underline{V} can be identified as a 3-representation of $\mathcal{Q}_{(Q'', Q, Q')}$. This clearly gives rise to the following proposition.

Proof. This follows from the construction of $\mathcal{Q}_{(Q,Q')}$ in Definition ((5.1)). In fact, there is a decomposing functor $\mathcal{F} : Rep_{(Q,Q')} \rightarrow Rep_k(\mathcal{Q}_{(Q,Q')})$ which identify any representation of $\mathcal{Q}_{(Q,Q')}$ as Q , Q' and a collection of compatibility maps and assign it to a birepresentation of (Q, Q') . Its inverse is the gluing functor $\mathcal{G} : Rep_k(\mathcal{Q}_{(Q,Q')}) \rightarrow Rep_{(Q,Q')}$, which view each birepresentation as three parts (two representations of Q , Q' and a collection of compatibility maps) and assemble them for a representation of $\mathcal{Q}_{(Q,Q')}$.

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Using Induction and Remark (3.2), we have the following.

Proposition 5.5. *For any $n \geq 2$, there exists an equivalence of categories $Rep_{(Q_1, Q_2, \dots, Q_n)} \simeq Rep_k(\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)})$ that restricts to an equivalence $rep_{(Q_1, Q_2, \dots, Q_n)} \simeq rep_k(\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)})$.*

Proof.

□

The following is an immediate consequence of Proposition (5.4) and Proposition (2.4).

Proposition 5.6. *There exists an equivalence of categories $Mod k\mathcal{Q}_{(Q, Q')} \simeq Rep_{(Q, Q')}$ that restricts to an equivalence $mod k\mathcal{Q}_{(Q, Q')} \simeq rep_{(Q, Q')}$.*

Proof.

□

By Remark (3.2) and Propositions (5.5), (2.4), we have the following.

Proposition 5.7. *For any $n \geq 2$, there exists an equivalence of categories $Mod k\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)} \simeq Rep_{(Q_1, Q_2, \dots, Q_n)}$ that restricts to an equivalence $mod k\mathcal{Q}_{(Q_1, Q_2, \dots, Q_n)} \simeq rep_{(Q_1, Q_2, \dots, Q_n)}$.*

Proof.

□

We recall the definitions of a co-wellpowered category and a generating set for a category.

Let \mathfrak{E} be a class of all epimorphisms of a category \mathfrak{A} . Then \mathfrak{A} is called *co-wellpowered* provided that no \mathfrak{A} -object has a proper class of pairwise non-isomorphic quotients [4, p. 125]. In other words, for every object the quotients form a set [26, p. 92, 95]. We refer the reader to [4] basics on quotients and co-wellpowered categories.

Following [20, p. 127], a set \mathcal{G} of objects of the category \mathcal{C} is said to *generate* \mathcal{C} when any parallel pair $f, g : X \rightarrow Y$ of arrows of \mathcal{C} , $f \neq g$ implies that there is an $G \in \mathcal{G}$ and an arrow $\alpha : G \rightarrow X$ in \mathcal{C} with $f\alpha \neq g\alpha$ (the term “generates” is well established but poorly chosen; “*separates*” would have been better). For the basic concepts of generating sets, we refer to [20], [4], or [15].

The following proposition immediately follows from Remark (3.2), Proposition (5.7) and the fact that the categories of modules are co-wellpowered with generating sets.

Proposition 5.8. *For any $n \geq 2$, the category $Rep_{(Q_1, Q_2, \dots, Q_n)}$ is co-wellpowered.*

Proof.

□

Proposition 5.9. *For any $n \geq 2$, the category $Rep_{(Q_1, Q_2, \dots, Q_n)}$ has a generating set.*

Proof.

□

Using Theorem (2.5), we have the following.

Proposition 5.10. *For any $n \geq 2$, the obvious forgetful functor $\mathcal{U}_n : \text{Rep}_{(Q_1, Q_2, \dots, Q_n)} \rightarrow \text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \times \dots \times \text{Rep}_k(Q_n)$ has a right adjoint. Equivalently, for any $n \geq 2$, the concrete category $(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}, \mathcal{U}_n)$ has cofree objects.*

Proof.

□

Let $\text{CoAlg}(\text{Rep}_{(Q, Q')})$ be the category of coalgebras in $\text{Rep}_{(Q, Q')}$. By [2, p. 30], the left $k\mathcal{Q}_{(Q, Q')}$ -module coalgebras which are finitely generated as left $k\mathcal{Q}_{(Q, Q')}$ -modules form a system of generators for $\text{CoAlg}(\text{Mod } k\mathcal{Q}_{(Q, Q')})$. Thus, from Proposition (2.2) and Theorem (2.5), we have the following.

Proposition 5.11. *The forgetful functor $\mathcal{U} : \text{CoAlg}(\text{Rep}_{(Q, Q')}) \rightarrow \text{Rep}_{(Q, Q')}$ has a right adjoint. Equivalently, the concrete category $(\text{Rep}_{(Q, Q')}, \mathcal{U})$ has cofree objects.*

Proof.

□

Remark (3.2) implies the following consequence.

Proposition 5.12. *For any $n \geq 2$, the forgetful functor $\mathcal{U}_n : \text{CoAlg}(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}) \rightarrow \text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$ has a right adjoint. Equivalently, for any $n \geq 2$, the concrete category $(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}, \mathcal{U}_n)$ has cofree objects.*

Proof.

□

Remark 5.13. If the forgetful functor $\mathcal{U} : \text{CoAlg}(\text{Rep}_{(Q, Q')}) \rightarrow \text{Rep}_{(Q, Q')}$ has a right adjoint \mathcal{V} and $\bar{M} \in \text{Rep}_{(Q, Q')}$, then the cofree coalgebra over \bar{M} can be given by

$$\mathcal{V}(\bar{M}) = \lim_{[\bar{f}: \bar{G} \rightarrow \bar{M}] \in \text{Rep}_{(Q, Q')}, \bar{G} \in \text{CoAlg}(\text{Rep}_{(Q, Q')}), \bar{G} \in \text{rep}_{(Q, Q')}} \bar{G}.$$

Thus, the concrete category $(\text{CoAlg}(\text{Rep}_{(Q, Q')}), \mathcal{U})$ has cofree objects given in terms of colimits and generators.

It follows that, for any $n \geq 2$, if the forgetful functor $\mathcal{U}_n : \text{CoAlg}(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}) \rightarrow \text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$ has a right adjoint \mathcal{V}_n and $\bar{M} \in \text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$, then the cofree coalgebra over \bar{M} can be given by

$$\mathcal{V}_n(\bar{M}) = \lim_{[\bar{f}: \bar{G} \rightarrow \bar{M}] \in \text{Rep}_{(Q_1, Q_2, \dots, Q_n)}, \bar{G} \in \text{CoAlg}(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}), \bar{G} \in \text{rep}_{(Q_1, Q_2, \dots, Q_n)}} \bar{G}.$$

The following proposition explicitly describes colimits and cofree objects in the product categories.

Proposition 5.14. *For any $m \in \{1, 2, 3, \dots, n\}$, let $\mathcal{U}_m : \mathcal{A}_m \rightarrow \mathcal{X}_m$ be a forgetful functor with a right adjoint \mathcal{V}_m . Then the product functor $\mathcal{V}_1 \times \mathcal{V}_2 \times \dots \times \mathcal{V}_n : \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n \rightarrow \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n$*

is a right adjoint of the obvious forgetful functor $\mathcal{U}_1 \times \mathcal{U}_2 \times \dots \times \mathcal{U}_n : \mathcal{A}_1 \times \mathcal{A}_2 \times \dots \times \mathcal{A}_n \rightarrow \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$. Furthermore, for any $(X_1, X_2, \dots, X_n) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$, the cofree object over (X_1, X_2, \dots, X_n) is exactly $(\mathcal{V}_1(X_1) \times \mathcal{V}_2(X_2) \times \dots \times \mathcal{V}_n(X_n))$.

Proof. The proof is straightforward. □

For any $m \in \{1, 2, \dots, n\}$, let $\mathcal{U}_m : \text{CoAlg}(\text{Rep}_k(Q_m)) \rightarrow \text{Rep}_k(Q_m)$ be the obvious forgetful functor with a right adjoint \mathcal{V}_m . Proposition (5.14) and Remark (3.2) suggest the following immediate consequence.

Corollary 5.15. *The product functor $\mathcal{V}_1 \times \mathcal{V}_2 \times \dots \times \mathcal{V}_n : \text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \times \dots \times \text{Rep}_k(Q_n) \rightarrow \text{CoAlg}(\text{Rep}_k(Q_1)) \times \text{CoAlg}(\text{Rep}_k(Q_2)) \times \dots \times \text{CoAlg}(\text{Rep}_k(Q_n))$ is a right adjoint of the obvious product forgetful functor $\mathcal{U}_1 \times \mathcal{U}_2 \times \dots \times \mathcal{U}_n : \text{CoAlg}(\text{Rep}_k(Q_1)) \times \text{CoAlg}(\text{Rep}_k(Q_2)) \times \dots \times \text{CoAlg}(\text{Rep}_k(Q_n)) \rightarrow \text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \times \dots \times \text{Rep}_k(Q_n)$. Furthermore, for any $(M_1, M_2, \dots, M_n) \in \text{Rep}_k(Q_1) \times \text{Rep}_k(Q_2) \times \dots \times \text{Rep}_k(Q_n)$, the cofree object over (M_1, M_2, \dots, M_n) is exactly $(\mathcal{V}_1(M_1), \mathcal{V}_2(M_2), \dots, \mathcal{V}_n(M_n))$.*

Proof. □

For any $m \in \{1, 2, \dots, n\}$, let $\mathcal{U}_m : \text{CoAlg}(\text{Rep}_k(Q_m)) \rightarrow \text{Rep}_k(Q_m)$ be the obvious forgetful functor with a right adjoint \mathcal{V}_m . By Proposition (5.9), for any $n \geq 2$, the category $\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$ has a generating set \mathcal{G} . For any $m \in \{1, 2, 3, \dots, n\}$, let \mathcal{G}_m be a generating set of $\text{Rep}_k(Q_m)$. The definition of n -representations of quivers implies that each element of \mathcal{G} takes the form $(G_1, G_2, \dots, G_n, \chi_1, \chi_2, \dots, \chi_{n-1})$ where $G_m \in \mathcal{G}_m$ for every $m \in \{1, 2, 3, \dots, n\}$. Furthermore, Remark (5.13) and Proposition (4.3) imply the following proposition which gives an explicit description for cofree coalgebras in the concrete category $(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}, \mathcal{U})$, where $\mathcal{U} : \text{CoAlg}(\text{Rep}_{(Q_1, Q_2, \dots, Q_n)}) \rightarrow \text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$ is the obvious forgetful functor with a right adjoint \mathcal{V} .

Proposition 5.16. *Let $\bar{M} = (M^{(1)}, M^{(2)}, \dots, M^{(n)}, \psi_1, \psi_2, \dots, \psi_{n-1}) \in \text{Rep}_{(Q_1, Q_2, \dots, Q_n)}$. Then the cofree object over \bar{M} is exactly $(\mathcal{V}_1(M^{(1)}), \mathcal{V}_2(M^{(2)}), \dots, \mathcal{V}_n(M^{(n)}), \psi'_1, \psi'_2, \dots, \psi'_{n-1})$, for some unique k -linear maps $\psi'_1, \psi'_2, \dots, \psi'_{n-1}$.*

Proof. This can be proved by applying Remark (5.13) and Proposition (4.3). Explicitly, if $\bar{G} = (G^{(1)}, G^{(2)}, \dots, G^{(n)}, \chi_1, \chi_2, \dots, \chi_{n-1})$, $\bar{f} = (f^{(1)}, f^{(2)}, \dots, f^{(n)})$, we have the following.

for some unique k -linear maps $\psi'_1, \psi'_2, \dots, \psi'_{n-1}$.

We end this paper by pointing out that the universal investigation above can be adjusted to study cofree objects in the centralizer and the center categories of $Rep_{(Q_1, Q_2, \dots, Q_n)}$. Indeed, one can use [1] to study and describe them in terms of cofree objects in centralizer and center categories of $Rep_k(Q_j)$, $j \in \{1, 2, \dots, n\}$.

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- [1] Adnan H. Abdulwahid. *Cofree Objects in The Centralizer and The Center Categories*. (2016). Preprint available at [ArXiv.org:math.CT/1603.02386](http://arxiv.org/math.CT/1603.02386). <http://arxiv.org/pdf/1603.02386v3.pdf>.
- [2] Adnan H. Abdulwahid, Miodrag C. Iovanov. *Generators for Comonoids and Universal Constructions*. Arch. Math. 106 (2016), 21-33, © 2015 Springer International Publishing, published online December 11, 2015, DOI 10.1007/s00013-015-0826-6.
- [3] Adnan H. Abdulwahid. *n-representations of Quivers*. (2016). Preprint available at [ArXiv.org:math.RT/1604.05512](http://arxiv.org/math.RT/1604.05512). <http://arxiv.org/pdf/1604.05512v4.pdf>.
- [4] J. Adamek , H. Herrlich, G. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. Dover Publication. 2009.
- [5] Ibrahim Assem, Andrzej Skowronski, Daniel Simson. *Elements of the Representation Theory of Associative Algebras*. Volume 1: Techniques of Representation Theory. London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006.
- [6] M. Auslander, Idun Reiten, S.O. Smalø. *Representation Theory of Artin Algebras*. Cambridge Studies in Advanced Mathematics 36, Cambridge University Press, Cambridge. 1995.
- [7] Steve Awodey. *Category Theory*. Oxford University Press, 2nd ed., 2010.
- [8] B. Bakalov, A. Kirillov Jr.. *Lectures on Tensor Categories and Modular Functor*. AMS University Lecture Series, 2001, 221 pp.
- [9] Michael Barot. *Introduction to The Representation Theory of Algebras*. © Springer International Publishing Switzerland. 2015.
- [10] D. J. Benson. *Representations and cohomology I: Basic representation theory of finite groups and associative algebras*. Cambridge Stud. Adv. Math. 30, Cambridge University Press, 1991.

- [11] F. Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*. Cambridge University Press. 1994.
- [12] F. Borceux. *Handbook of Categorical Algebra 2: Categories and Structures*. Cambridge University Press. 1994.
- [13] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, Victor Ostrik. *Tensor Categories*. Mathematical Surveys and Monographs 205, American Mathematical Society, 2015.
- [14] Pavel Etingof, Oleg Golberg, Sebastian Hensel Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, Elena Yudovina. *Introduction to Representation Theory*. Volume 59, American Mathematical Society, RI, 2011.
- [15] P.J. Freyd, A. Scedrov. *Categories, Allegories*. Elsevier Science Publishing Company, Inc. 1990.
- [16] P. Gabriel. *Unzerlegbare Darstellungen I*. Manuscripta Math., 6 (1972), 71103.
- [17] Mati Kilp, Ulrich Knauer, Alexander V. Mikhalev . *Monoids, Acts, and Categories: With Applications to Wreath Products and Graphs*. De Gruyter exposition in math., Berlin 29. 2000.
- [18] Tom Leinster. *Basic Category Theory*. volume 298 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2014.
- [19] Tom Leinster. *Higher Operads, Higher Categories*. volume 298 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2004.
- [20] S. Mac Lane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics vol. 5, Springer-Verlag, New York, 2nd ed., 1998.
- [21] Colin McLarty. *Elementary Categories, Elementary Toposes*. Oxford University Press. New York. 2005.
- [22] B. Mitchell. *Theory of Categories*. Academic Press, 1965.
- [23] B. Pareigis. *Categories and Functors*. Academic Press, 1971.
- [24] Joseph J. Rotman. *An Introduction to Homological Algebra*. 2nd Edition, Springer, New York, 2009.
- [25] Ralf Schiffler. *Quiver Representations*. CMS Books in Mathematics Series. Springer International Publishing. 2014.
- [26] H. Schubert. *Categories*. Springer-Verlag, 1972.
- [27] V.V. Sergeichuk. *Linearization method in classification problems of linear algebra*. São Paulo J. Math. Sci.1(no. 2) (2007), 219240.
- [28] Ross Street. *Quantum Groups: a path to current algebra*. Number 19 in Australian Mathematical Society Lecture Series. Cambridge University Press, 2007.
- [29] R. Wisbauer. *Foundations of Module and Ring Theory: A Handbook for Study and Research*. Springer-Verlag. 1991.
- [30] Alexander Zimmermann. *Representation Theory: A Homological Algebra Point of View*. Springer Verlag London, 2014.

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